EINDHOVEN UNIVERSITY OF TECHNOLOGY

Symmetry Handling in SCIP

An Overview

Christopher Hojny

Let’s SCIP it!, Zuse Institute Berlin, November 4, 2022

20 years of SCIP – 5 years of symmetry handling

```
commit e45f24868cf7a18c79880afeef84af57902c5f83
Author: Marc Pfetsch <pfetsch@mathematik.tu-darmstadt.de>
Date:   Sat Sep 30 13:35:31 2017 +0200

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commit 8bcb00fab87f41d9b5a5614b72dd2f3ae900b998
Author: Marc Pfetsch <m.pfetsch@tu-bs.de>
Date:   Sat Sep 26 11:31:20 2009 +0000

    - first version (should work for partitioning)
```
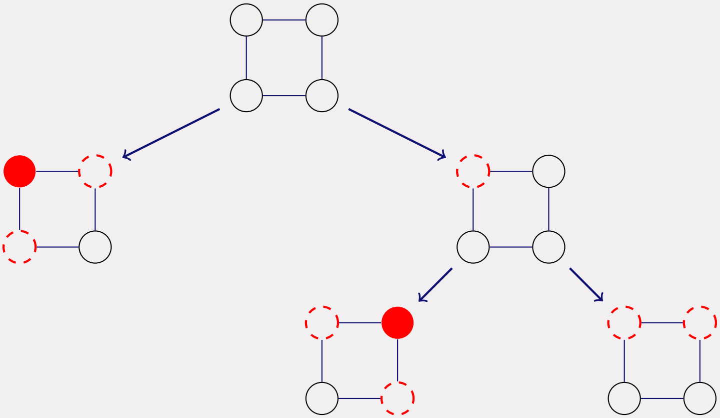

Symmetry in Branch-and-Bound



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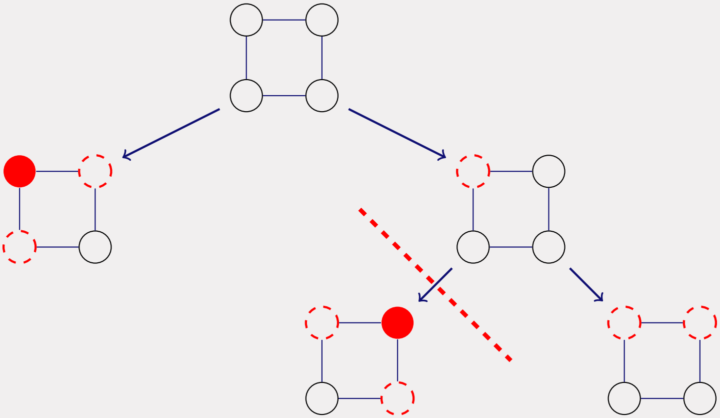

Outline

Symmetry Detection

General Variables

Binary Variables
- Orbital Fixing
- Symmetry Handling Constraints

User Interaction

Problem Setting

We consider optimization problems of type

$$\max\{c^\top x \ : \ f(x,y) \leq 0,\ (x,y) \in \mathbb{Z}^n \times \mathbb{R}^p\},$$

where $f \colon \mathbb{Z}^n \times \mathbb{R}^p \to \mathbb{R}^m$.

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Some Definitions

- The action of a permutation $\gamma \in \mathcal{S}_n \times \mathcal{S}_p$ on $(x,y) \in \mathbb{Z}^n \times \mathbb{R}^p$ is

$$\gamma(x,y) = (x_{\gamma^{-1}(1)}, \dots, x_{\gamma^{-1}(n)}, y_{\gamma^{-1}(1)}, \dots, y_{\gamma^{-1}(p)}).$$

- A permutation $\gamma \in \mathcal{S}_n \times \mathcal{S}_p$ is a symmetry if, for every $(x,y) \in \mathbb{Z}^n \times \mathbb{R}^p$,
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Bad News: Already for MIPs, finding all symmetries is NP-hard.

Formulation Symmetries (Salvagnin 2005)

Consider MIP $\max\{c^\top x \,:\, Ax \le b,\ x \in \mathbb{Z}^n\}$

$$
\begin{aligned}
\max\ & x_1 + x_2 \\
& x_1 \qquad + 2x_3 \le 3 \\
& \qquad x_2 + 2x_3 \le 3
\end{aligned}
$$

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$\gamma \in \mathcal{S}_n$ is a formulation symmetry if there exists a permutation π of the rows of (A, b) such that

- $\gamma(c) = c$,
- $\pi(b) = b$,
- $A_{\pi^{-1}(i),\gamma^{-1}(j)} = A_{i,j}$.

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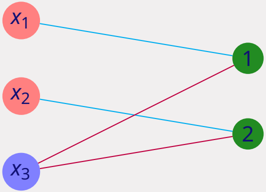

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Graph automorphism codes can be used to detect formulation symmetries.

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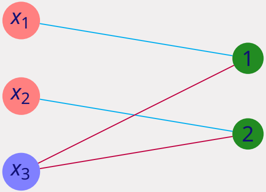

Implementation in SCIP

- different symmetry detection graphs for MIPs and MINLPs
- automorphism code `bliss`
- returns list of generators of symmetry group
- possibility to limit number of generators

```
(0.3s) symmetry computation started: requiring (bin +, int -, cont +), (fixed: bin -, int +, cont -)
(0.5s) symmetry computation finished: 9 generators found (max: 1500, log10 of symmetry group size: 6.6)
```

General Variables

Basic Idea to Handle Symmetries

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Handle symmetries by sorting solutions, discard solutions that are not maximal in this sorting.

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- select variable x_i
- compute the orbit $\mathrm{orbit}(\Gamma, i) = \{\gamma(i) : \gamma \in \Gamma\}$
- add inequalities

$$x_i \geq x_j, \qquad j \in \mathrm{orbit}(\Gamma, i)$$

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$$x_i \geq x_j, \qquad j \in \mathrm{orbit}(\Gamma, i)$$

- we call i the leader and j the follower of the cut
- **Pro:** very simple inequalities
- **Con:** amount of handled symmetries very limited

Cuts from Schreier-Sims Table (SST Cuts)

Improved Idea (Liberti and Ostrowski 2014, Salvagnin 2018)

Iteratively add the simple inequalities.

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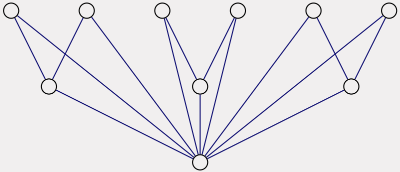

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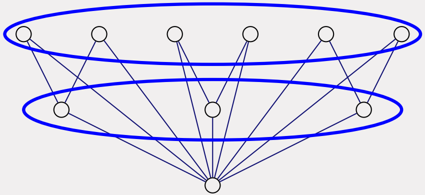

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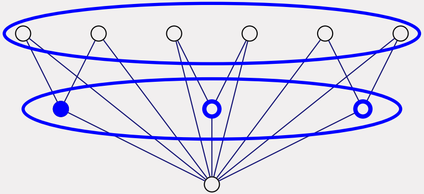

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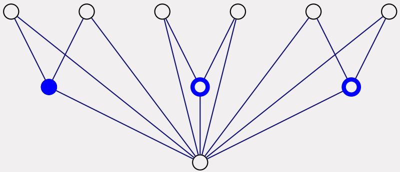

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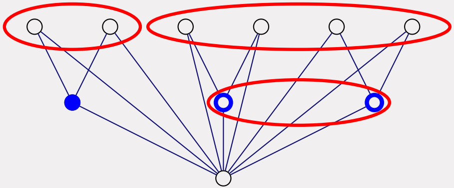

Parameterizing SST Cuts

many degrees of freedom in generating SST cuts, among others,

- type of variable x_i:
  - binary
  - general integer
  - continuous
- type of orbit:
  - orbit of maximum size
  - orbit of minimum size
  - orbit with most conflicts

Numerical Results I – Minimum Orbit

variable type						
bin	int	cont	orbit	symresack	time	#opt
					1107.4	126
x			min		-3.1 %	128
	x		min		±0 %	127
		x	min		±0 %	125
	x	x	min		±0 %	126
x	x	x	min		-1.5 %	127

Setup:

- MIP solver: SCIP 8.0.0.2
- LP solver: SoPlex 6.0.0.2
- test set: MIPLIB 2017 benchmark (240 instances)
- time limit: 2 h per instance

Numerical Results II – Maximum Orbit

variable type						
bin	int	cont	orbit	symresack	time	#opt
					1107.4	126
x			max		-7.0 %	130
	x		max		-0.3 %	127
		x	max		+0.2 %	125
	x	x	max		-0.2 %	126
x	x	x	max		-2.2 %	129

Numerical Results III – Maximum Conflict Orbit

variable type						
bin	int	cont	orbit	symresack	time	#opt
					1107.4	126
x			conf		-7.1 %	129
x	x	x	conf		-5.2 %	128

Binary Variables

Binary Programs

We consider

$$\max\{c^\top x \,:\, Ax \leq b,\ x \in \{0,1\}^n\}.$$

Assume that symmetry group Γ of binary program is known.

SCIP has two dedicated symmetry handling techniques for binary variables:

- orbital fixing
- symmetry handling constraints

Common Ground: for each set of equivalent solutions, it is sufficient to compute a representative solution

Graph Coloring

Input

- undirected graph $G = (V, E)$
- positive integer k

Task

find maximum induced subgraph of G that admits proper k-coloring

$$\max \sum_{v \in V} \sum_{j=1}^{k} x_{vj}$$

$$\begin{aligned} \sum_{j=1}^{k} x_{vj} &\leq 1, && v \in V, \\ x_{uj} + x_{vj} &\leq 1, && \{u, v\} \in E,\ j \in \{1, \dots, k\}, \\ x_{vj} &\in \{0, 1\}, && v \in V,\ j \in \{1, \dots, k\} \end{aligned}$$

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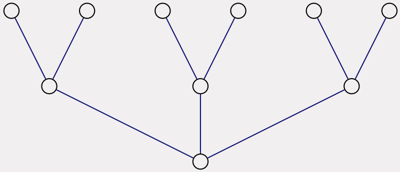
Symmetries

- color symmetries $\rightsquigarrow$ column permutations
- graph symmetries $\rightsquigarrow$ row permutations

Orbital Fixing (Margot 2003, Ostrowski 2009)

Idea: based on branching decisions, exclude non-representative solutions by fixing variables



Steps

1. $I \leftarrow$ all variables fixed to 1 by branching
2. compute $\Gamma' = \{\gamma \in \Gamma : \gamma(I) = I\}$
3. for each $i \in \{1, \dots, n\}$, compute $O = \{\gamma(i) : \gamma \in \Gamma\}$
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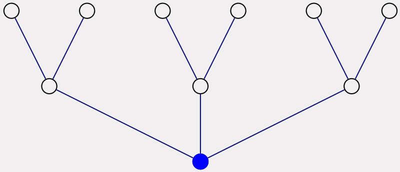

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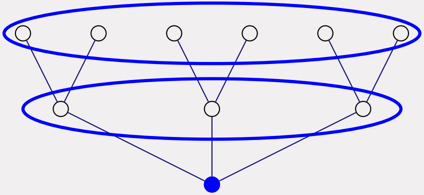

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Symmetry Handling Constraints

Main Idea: for each set of symmetric solutions, forbid solutions which are not lexicographically maximal

- Friedman 2007: for every $\gamma \in \Gamma$

$$\sum_{i=1}^{n} 2^{n-i} x_i \geq \sum_{i=1}^{n} 2^{n-i} \gamma(x)_i$$

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- SCIP's approach: consider symresack

$$P_\gamma = \operatorname{conv}\left\{x \in \{0,1\}^n \,:\, \sum_{i=1}^{n} 2^{n-i} x_i \geq \sum_{i=1}^{n} 2^{n-i} \gamma(x)_i\right\}$$

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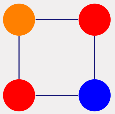
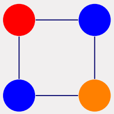
- cover inequalities can be separated in $O(n)$ time (H. and Pfetsch 2018)
- symresacks can be propagated in $O(n)$ time (van Doornmalen and H. 2022+)

Symmetric Groups

More reductions can be found by taking entire group Γ into account.

- if all variables can be sorted arbitrarily, use $x_1 \geq x_2 \geq \cdots \geq x_n$
- in graph coloring, we can sort groups (columns) of variables arbitrarily
- orbitopes sort columns of binary matrices lexicographically non-increasingly





$$\begin{pmatrix} 1 & & \\ & 1 & \\ 1 & & \\ & & 1 \end{pmatrix} \qquad \begin{pmatrix} & 1 & \\ & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$$

Orbitopes

Idea: sort columns lexicographically non-increasingly

Full Orbitopes

- no further restriction on binary matrices
- linear time propagation algorithm (Bendotti et al. 2019
- linear time separation algorithm for IP formulation (H. and Pfetsch 2018)

Packing/Partitioning Orbitopes

- each row has at most/exactly one 1-entry
- linear time propagation algorithm (Kaibel et al. 2011)
- linear time separation algorithm of facet description (Kaibel and Pfetsch 2008)

Symmetry Handling Constraints in SCIP

split symmetry group Γ into independent factors $\Gamma = \Gamma_1 \otimes \cdots \otimes \Gamma_\ell$

- orbitope detection:
  - heuristic to detect whether Γ_i can be completely handled by orbitopes
  - decide whether full or packing/partitioning orbitopes are used
- no orbitope is added?
  - scan list of generators of Γ_i
  - try to find "hidden" orbitopes
  - handle remaining generators by symresacks
  - possibly add SST cuts
- no hidden orbitopes found?
  - use orbital fixing

SCIP 8.0 $\rightsquigarrow$ running time improvement on MIPLIB 2017 benchmark testset: 16 %

User Interaction

Providing Symmetries

- providing list of symmetries currently not possible
- symmetries can be provided by three types of constraint handlers
  - symresacks enforce x is not lex. smaller than $\gamma(x)$

    symresack([x_1,...,x_n],[$\gamma(1)$,...,$\gamma(n)$]);

  - orbitopes sort columns of matrices $X \in \{0,1\}^{m \times n}$

    fullOrbitope($x_{1,1}$,...,$x_{1,n}$. ... .x_{m1},...,x_{mn});

    packOrbitope(...); partOrbitope(...);

  - orbisacks are orbitopes with two columns

    fullOrbisack(...); packOrbisack(...); partOrbisack(...);

Summary

SCIP's symmetry handling approach

- detect symmetries by detecting symmetries of auxiliary colored graph
- mixed symmetry handling strategy
  - SST cuts
  - orbitopes und symresacks
  - orbital fixing

TU/e

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