UNIVERSITY OF TWENTE.

Simple odd β-cycle inequalities for binary polynomial optimization

Matthias Walter (Uni Twente)

Joint work with Alberto del Pia (Uni Wisconsin-Madison)

Let's SCIP it! A workshop to celebrate 20 years of SCIP, 2022

UNIVERSITY OF TWENTE.

Simple odd β-cycle inequalities for binary polynomial optimization

Matthias Walter (Uni Twente)

Joint work with Alberto del Pia (Uni Wisconsin-Madison)

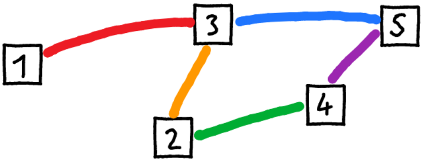
Let's SCIP it! A workshop to celebrate 20 years of SCIP, 2022

Definition – Boolean quadric polytope **[Padberg '88]**

Let $G = (V, E)$ be a graph. The **boolean quadric polytope** of G is the polytope

$$\text{BQP}(G) := \text{conv}\left\{(x,y) \in \{0,1\}^V \times \{0,1\}^E : y_{\{u,v\}} = x_u \cdot x_v \text{ for each edge } \{u,v\} \in E\right\}.$$

Example:



$y_{\{1,3\}} = x_1 \cdot x_3$

$y_{\{2,3\}} = x_2 \cdot x_3$

$y_{\{2,4\}} = x_2 \cdot x_4$

$y_{\{3,5\}} = x_3 \cdot x_5$

$y_{\{4,5\}} = x_4 \cdot x_5$

$x_i \in \{0,1\} \quad i = 1, \dots, 5$

Remarks:

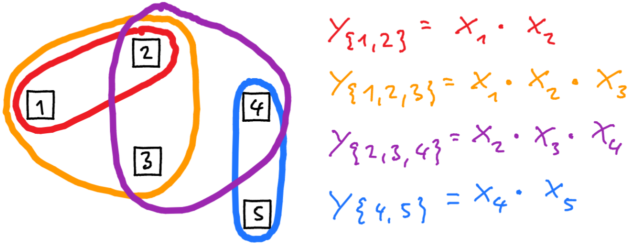
- Equivalent to CUT polytope of related graph. [Barahona, Mahjoub '86]
- Can be used to minimize a quadratic function $q(x)$ over $x \in \{0,1\}^n$, also known as "quadratic unconstrained binary optimization".
- Optimization over BQP is NP-hard in general.

Definition – Multilinear polytope [Del Pia, Khajavirad '16; Buchheim, Crama, Rodríguez-Heck '16]

Let $G = (V, E)$ be a hypergraph. The **multilinear polytope** of G is the polytope

$$\text{ML}(G) := \text{conv}\left\{(x,y) \in \{0,1\}^V \times \{0,1\}^E : y_e = \prod_{v \in e} x_v \text{ for each edge } \{u,v\} \in E\right\}.$$

Example:



Remarks:

- Can be used to minimize a polynomial function $p(x)$ over $x \in \{0,1\}^n$, also known as "polynomial unconstrained binary optimization" or "pseudo-boolean optimization".
- For each hyperedge $e = \{v_1, v_2, \ldots, v_k\}$, we have the logic AND constraint $y_e = x_{v_1} \wedge x_{v_2} \wedge \cdots \wedge x_{v_k}$.

Standard Relaxation

Proposition – Standard relaxation [Fortet '60; Glover, Woolsey '74]

Let $G = (V, E)$ be a hypergraph. The polytope $\mathrm{SR}(G)$ defined by

$$0 \leq y_e \leq x_v \leq 1 \qquad \forall v \in e \in E \qquad (1a)$$

$$y_e + \sum_{v \in e} (1 - x_v) \geq 1 \qquad \forall e \in E \qquad (1b)$$

yields an IP formulation, i.e., $\mathrm{SR}(G) \cap \mathbb{Z}^{V+E} = \mathrm{ML}(G) \cap \mathbb{Z}^{V+E}$.

Standard Relaxation

Proposition – Standard relaxation [Fortet '60; Glover, Woolsey '74]

Let $G = (V, E)$ be a hypergraph. The polytope $\mathrm{SR}(G)$ defined by

$$0 \leq y_e \leq x_v \leq 1 \qquad \forall v \in e \in E \qquad (1a)$$

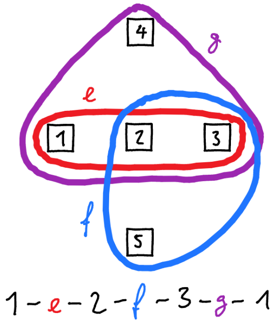
$$y_e + \sum_{v \in e} (1 - x_v) \geq 1 \qquad \forall e \in E \qquad (1b)$$

yields an IP formulation, i.e., $\mathrm{SR}(G) \cap \mathbb{Z}^{V+E} = \mathrm{ML}(G) \cap \mathbb{Z}^{V+E}$.

Berge cycle:

$v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_1$ with:

- $v_i \in V$ are distinct nodes.
- $e_i \in E$ are distinct edges.
- $v_i \in e_{i-1} \cap e_i$ for each i



Standard Relaxation

Proposition – Standard relaxation [Fortet '60; Glover, Woolsey '74]

Let $G = (V, E)$ be a hypergraph. The polytope $\mathrm{SR}(G)$ defined by

$$0 \leq y_e \leq x_v \leq 1 \qquad \forall v \in e \in E \qquad (1a)$$

$$y_e + \sum_{v \in e} (1 - x_v) \geq 1 \qquad \forall e \in E \qquad (1b)$$

yields an IP formulation, i.e., $\mathrm{SR}(G) \cap \mathbb{Z}^{V+E} = \mathrm{ML}(G) \cap \mathbb{Z}^{V+E}$.

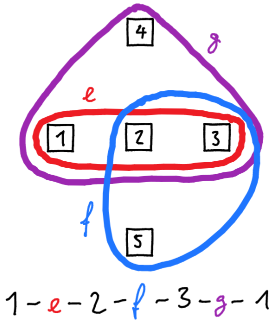
Theorem – Perfect formulation [Del Pia, Khajavirad '16; Buchheim, Crama, Rodríguez-Heck '16]

$\mathrm{SR}(G) = \mathrm{ML}(G)$ holds if and only if G is **Berge-acyclic**.

Berge cycle:

$v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_1$ with:

- $v_i \in V$ are distinct nodes.
- $e_i \in E$ are distinct edges.
- $v_i \in e_{i-1} \cap e_i$ for each i



Standard Relaxation

Proposition – Standard relaxation [Fortet '60; Glover, Woolsey '74]

Let $G = (V, E)$ be a hypergraph. The polytope SR(G) defined by

$$0 \leq y_e \leq x_v \leq 1 \qquad \forall v \in e \in E \qquad (1a)$$

$$y_e + \sum_{v \in e} (1 - x_v) \geq 1 \qquad \forall e \in E \qquad (1b)$$

yields an IP formulation, i.e., $\text{SR}(G) \cap \mathbb{Z}^{V+E} = \text{ML}(G) \cap \mathbb{Z}^{V+E}$.

Theorem – Perfect formulation [Del Pia, Khajavirad '16; Buchheim, Crama, Rodríguez-Heck '16]

$\text{SR}(G) = \text{ML}(G)$ holds if and only if G is **Berge-acyclic**.

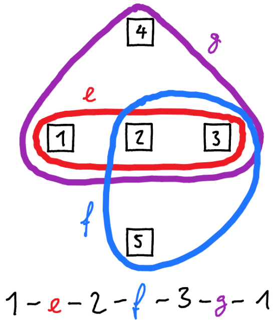
Remark:

- There exist several definitions of cycles in hypergraphs, such as Berge cycles, β-cycles, γ-cycles.

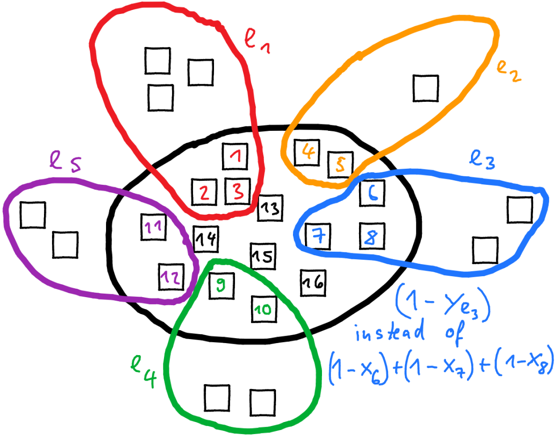
Berge cycle:

$v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_1$ with:

- $v_i \in V$ are distinct nodes.
- $e_i \in E$ are distinct edges.
- $v_i \in e_{i-1} \cap e_i$ for each i



Flower Relaxation



Definition – Flower relaxation [Del Pia, Khajavirad '18]

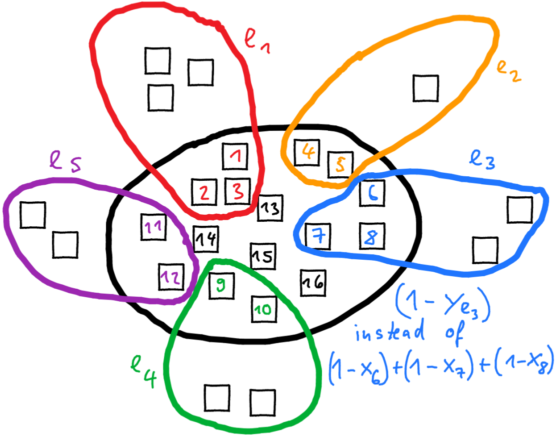
The k-**flower inequality centered at** f **with neighbors** $e_1, e_2, \dots, e_k$ is

$$y_f + \sum_{i=1}^{k}(1 - y_{e_i}) + \sum_{v \in R}(1 - x_v) \geq 1, \qquad (2)$$

where $R := f \setminus \bigcup_{i=1}^{k} e_i$ contains all nodes of v that are not in a leaf. We denote by FR(G) the standard relaxation SR(G), augmented by all 1-flower and all 2-flower inequalities.

For comparison: $y_f + \sum_{v \in f}(1 - x_v) \geq 1 \qquad (1b)$

Flower Relaxation



Definition – Flower relaxation [Del Pia, Khajavirad '18]

The k-**flower inequality centered at** f **with neighbors** $e_1, e_2, \ldots, e_k$ is

$$y_f + \sum_{i=1}^{k}(1 - y_{e_i}) + \sum_{v \in R}(1 - x_v) \geq 1, \qquad (2)$$

where $R := f \setminus \bigcup_{i=1}^{k} e_i$ contains all nodes of v that are not in a leaf. We denote by FR(G) the standard relaxation SR(G), augmented by all 1-flower and all 2-flower inequalities.

For comparison: $y_f + \sum_{v \in f}(1 - x_v) \geq 1 \qquad (1b)$

Special case and generalization:

- 1-flower inequalities were independently introduced as **2-link inequalities**. [Crama, Rodríguez-Heck '17]
- Flower inequalities can be generalized to **running intersection inequalities**. [Del Pia, Khajavirad '21]

Odd β-Cycle Inequalities

Definition – Odd β-cycle inequalities [Del Pia, Di Gregorio '19]

Definition 2. Consider a hypergraph $G = (V, E)$, let $C = v_1, e_1, v_2, \dots, v_m, e_m, v_1$ be a β-cycle in G, and let E^-, E^+ be a partition of $E(C)$ such that $k := |E^-|$ is odd and $e_1 \in E^-$. Let $D := \{e_{p+1}, e_{p+2}, \dots, e_m\}$, where e_p is the last edge in C that belongs to E^-. We denote by $f_1, \dots, f_k$ the subsequence of $e_1, \dots, e_m$ of the edges in E^-. Let $S_1 := (\cup_{e \in E^-} e) \setminus \bigcup_{e \in E^+} e$ and $S_2 := V(C) \setminus \bigcup_{e \in E^-} e$. With this notation in place, we make the following assumptions:

(a) Every node $v \in \bigcup_{i=1}^m e_i$ is contained in at most two edges among $e_1, \dots, e_m$.

(b) For every edge $e_i \in E^+ \setminus D$, every edge in E^- adjacent to e_i (if any) is either e_{i-1} or e_{i+1}.

(c) No edge in D is adjacent to an edge f_i with i even.

(d) At least one of the following two conditions holds:

(d1) For every $v \in S_1$, either v is contained in just one edge $e \in E^-$, or it is contained in two edges f_i, f_j with i odd and j even.

(d2) For every $e' \in E^-$ and $e'' \in D$ such that $e' \cap e'' \neq \emptyset$, then either $e' = e_1$, $e'' = e_m$ or $e' = e_p$, $e'' = e_{p+1}$.

We then define the odd β-cycle inequality corresponding to C and E^- as

$$\sum_{v \in S_1} z_v - \sum_{e \in E^-} z_e - \sum_{v \in S_2} z_v + \sum_{e \in E^+} z_e \leq |S_1| - |\{i \in \{1, \dots, m\} : e_i, e_{i+1} \in E^-\}| + \left\lfloor \frac{k}{2} \right\rfloor. \tag{2}$$

Definition – Odd β-cycle inequalities [Del Pia, Di Gregorio '19]

Definition 2. Consider a hypergraph $G = (V, E)$, let $C = v_1, e_1, v_2, \dots, v_m, e_m, v_1$ be a β-cycle in G, and let E^-, E^+ be a partition of $E(C)$ such that $k := |E^-|$ is odd and $e_1 \in E^-$. Let $D := \{e_{p+1}, e_{p+2}, \dots, e_m\}$, where e_p is the last edge in C that belongs to E^-. We denote by $f_1, \dots, f_k$ the subsequence of $e_1, \dots, e_m$ of the edges in E^-. Let $S_1 := (\cup_{e \in E^-} e) \setminus \bigcup_{e \in E^+} e$ and $S_2 := V(C) \setminus \bigcup_{e \in E^-} e$. With this notation in place, we make the following assumptions:

(a) Every node $v \in \bigcup_{i=1}^m e_i$ is contained in at most two edges among $e_1, \dots, e_m$.

(b) For every edge $e_i \in E^+ \setminus D$, every edge in E^- adjacent to e_i (if any) is either e_{i-1} or e_{i+1}.

(c) No edge in D is adjacent to an edge f_i with i even.

(d) At least one of the following two conditions holds:

(d1) For every $v \in S_1$, either v is contained in just one edge $e \in E^-$, or it is contained in two edges f_i, f_j with i odd and j even.

(d2) For every $e' \in E^-$ and $e'' \in D$ such that $e' \cap e'' \neq \emptyset$, then either $e' = e_1$, $e'' = e_m$ or $e' = e_p$, $e'' = e_{p+1}$.

We then define the odd β-cycle inequality corresponding to C and E^- as

$$\sum_{v \in S_1} z_v - \sum_{e \in E^-} z_e - \sum_{v \in S_2} z_v + \sum_{e \in E^+} z_e \leq |S_1| - |\{i \in \{1, \dots, m\} : e_i, e_{i+1} \in E^-\}| + \left\lfloor \frac{k}{2} \right\rfloor. \tag{2}$$

Odd β-Cycle Inequalities

Definition – Odd β-cycle inequalities [Del Pia, Di Gregorio '19]

Definition 2. Consider a hypergraph $G = (V, E)$, let $C = v_1, e_1, v_2, \ldots, v_m, e_m, v_1$ be a β-cycle in G, and let E^-, E^+ be a partition of $E(C)$ such that $k := |E^-|$ is odd and $e_1 \in E^-$. Let $D := \{e_{p+1}, e_{p+2}, \ldots, e_m\}$, where e_p is the last edge in C that belongs to E^-. We denote by $f_1, \ldots, f_k$ the subsequence of $e_1, \ldots, e_m$ of the edges in E^-. Let $S_1 := (\cup_{e \in E^-} e) \setminus \bigcup_{e \in E^+} e$ and $S_2 := V(C) \setminus \bigcup_{e \in E^-} e$. With this notation in place, we make the following assumptions:

(a) Every node $v \in \bigcup_{i=1}^m e_i$ is contained in at most two edges among $e_1, \ldots, e_m$.

(b) For every edge $e_i \in E^+ \setminus D$, every edge in E^- adjacent to e_i (if any) is either e_{i-1} or e_{i+1}.

(c) No edge in D is adjacent to an edge f_i with i even.

(d) At least one of the following two conditions holds:

(d1) For every $v \in S_1$, either v is contained in just one edge $e \in E^-$, or it is contained in two edges f_i, f_j with i odd and j even.

(d2) For every $e' \in E^-$ and $e'' \in D$ such that $e' \cap e'' \neq \emptyset$, then either $e' = e_1$, $e'' = e_m$ or $e' = e_p$, $e'' = e_{p+1}$.

We then define the odd β-cycle inequality corresponding to C and E^- as

$$\sum_{v \in S_1} z_v - \sum_{e \in E^-} z_e - \sum_{v \in S_2} z_v + \sum_{e \in E^+} z_e \leq |S_1| - |\{i \in \{1, \ldots, m\} : e_i, e_{i+1} \in E^-\}| + \left\lfloor \frac{k}{2} \right\rfloor. \quad (2)$$

Odd β-Cycle Inequalities

Definition – Odd β-cycle inequalities [Del Pia, Di Gregorio '19]

Definition 2. Consider a hypergraph $G = (V, E)$, let $C = v_1, e_1, v_2, \ldots, v_m, e_m, v_1$ be a β-cycle in G, and let E^-, E^+ be a partition of $E(C)$ such that $k := |E^-|$ is odd and $e_1 \in E^-$. Let $D := \{e_{p+1}, e_{p+2}, \ldots, e_m\}$, where e_p is the last edge in C that belongs to E^-. We denote by $f_1, \ldots, f_k$ the subsequence of $e_1, \ldots, e_m$ of the edges in E^-. Let $S_1 := (\cup_{e \in E^-} e) \setminus \bigcup_{e \in E^+} e$ and $S_2 := V(C) \setminus \bigcup_{e \in E^-} e$. With this notation in place, we make the following assumptions:

(a) Every node $v \in \bigcup_{i=1}^m e_i$ is contained in at most two edges among $e_1, \ldots, e_m$.
(b) For every edge $e_i \in E^+ \setminus D$, every edge in E^- adjacent to e_i (if any) is either e_{i-1} or e_{i+1}.
(c) No edge in D is adjacent to an edge f_i with i even.
(d) At least one of the following two conditions holds:
(d1) For every $v \in S_1$, either v is contained in just one edge $e \in E^-$, or it is contained in two edges f_i, f_j with i odd and j even.
(d2) For every $e' \in E^-$ and $e'' \in D$ such that $e' \cap e'' \neq \emptyset$, then either $e' = e_1$, $e'' = e_m$ or $e' = e_p$, $e'' = e_{p+1}$.

We then define the odd β-cycle inequality corresponding to C and E^- as

$$\sum_{v \in S_1} z_v - \sum_{e \in E^-} z_e - \sum_{v \in S_2} z_v + \sum_{e \in E^+} z_e \leq |S_1| - |\{i \in \{1, \ldots, m\} : e_i, e_{i+1} \in E^-\}| + \left\lfloor \frac{k}{2} \right\rfloor. \tag{2}$$

Odd β-Cycle Inequalities

Definition – Odd β-cycle inequalities **[Del Pia, Di Gregorio '19]**

Let $G = (V, E)$ be a hypergraph. If there is a β-cycle C with a certain edge bipartition and some extra definitions satisfying some extra properties, then

$$\langle \text{some inequality with complicated coefficients and complicated right-hand side} \rangle$$

is called **odd β-cycle inequality** and valid for $\mathrm{ML}(G)$.

Odd β-Cycle Inequalities

Definition – Odd β-cycle inequalities **[Del Pia, Di Gregorio '19]**

Let $G = (V, E)$ be a hypergraph. If there is a β-cycle C with a certain edge bipartition and some extra definitions satisfying some extra properties, then

$$\langle \text{some inequality with complicated coefficients and complicated right-hand side} \rangle$$

is called **odd β-cycle inequality** and valid for $\text{ML}(G)$.

Theorem – CG rank **[Del Pia, Di Gregorio '19]**

Odd β-cycle inequalities have Chvátal rank 2 (w.r.t. SR).

Odd β-Cycle Inequalities

Definition – Odd β-cycle inequalities **[Del Pia, Di Gregorio '19]**

Let $G = (V, E)$ be a hypergraph. If there is a β-cycle C with a certain edge bipartition and some extra definitions satisfying some extra properties, then

$$\langle \text{some inequality with complicated coefficients and complicated right-hand side} \rangle$$

is called **odd β-cycle inequality** and valid for $\text{ML}(G)$.

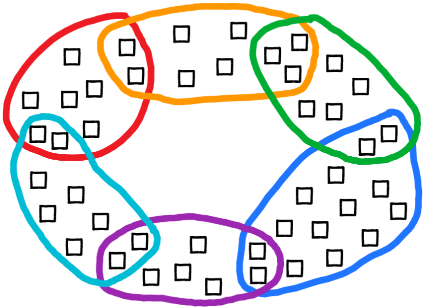
Theorem – CG rank **[Del Pia, Di Gregorio '19]**

Odd β-cycle inequalities have Chvátal rank 2 (w.r.t. SR).

Theorem – Perfection **[Del Pia, Di Gregorio '19]**

For **cyclic hypergraphs** G, $\text{ML}(G)$ is completely described by $\text{FR}(G)$ plus odd β-cycle inequalities.

Cyclic hypergraph:



Odd β-Cycle Inequalities

Definition – Odd β-cycle inequalities [Del Pia, Di Gregorio '19]

Let $G = (V, E)$ be a hypergraph. If there is a β-cycle C with a certain edge bipartition and some extra definitions satisfying some extra properties, then

$$\langle \text{some inequality with complicated coefficients and complicated right-hand side} \rangle$$

is called **odd β-cycle inequality** and valid for $\mathrm{ML}(G)$.

Theorem – CG rank [Del Pia, Di Gregorio '19]

Odd β-cycle inequalities have Chvátal rank 2 (w.r.t. SR).

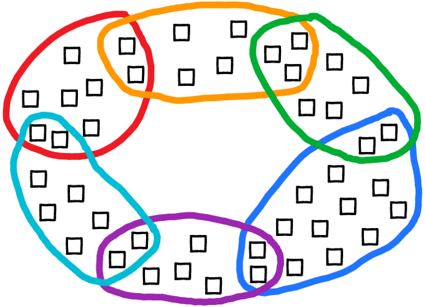
Theorem – Perfection [Del Pia, Di Gregorio '19]

For **cyclic hypergraphs** G, $\mathrm{ML}(G)$ is completely described by $\mathrm{FR}(G)$ plus odd β-cycle inequalities.

Theorem – Separation [Del Pia, Di Gregorio '19]

For **cyclic hypergraphs**, the separation problem for odd β-cycle inequalities can be solved in polynomial time.

Cyclic hypergraph:



Simple Odd β-Cycle Inequalities

Definition – Simple Odd β-cycle inequalities **[Del Pia, Walter '22]**

(see next slides)

Remark:

- The new definition yields weaker inequalities in general!

Theorem – CG rank **[Del Pia, Walter '22]**

Simple odd β-cycle inequalities have Chvátal rank 2 (with respect to the standard relaxation SR).

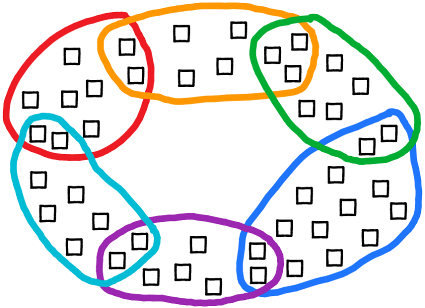
Theorem – Perfection **[Del Pia, Walter '22]**

For **cyclic hypergraphs** G, $\mathrm{ML}(G)$ is completely described by $\mathrm{FR}(G)$ plus **simple** odd β-cycle inequalities.

Theorem – Separation **[Del Pia, Walter '22]**

For **arbitrary hypergraphs**, the separation problem for **simple** odd β-cycle inequalities can be solved in polynomial time.

Cyclic hypergraph:



Building Block Inequalities

We consider an edge sequence e, f, g with $U = e \cap f$, $W = f \cap g$:

Lemma – Building block inequalities

The following **building block inequalities** $s(x,y) \geq 0$ are valid for $\mathrm{FR}(G)$.

$$2y_f - 1 + \sum_{u \in U}(1 - x_u) + \sum_{w \in W}(1 - x_w) + \sum_{v \in f \setminus (U \cup W)} (2 - 2x_v) \geq 0$$

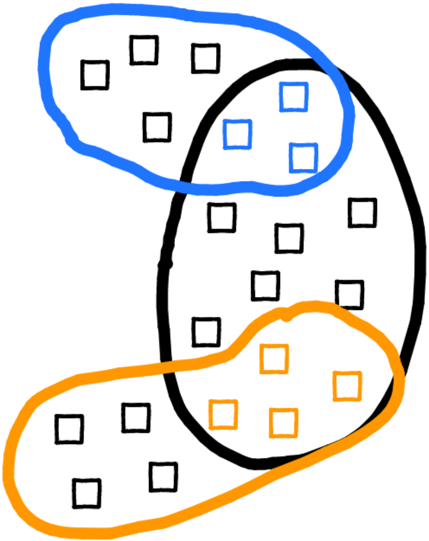
$$2y_f - 1 + (1 - y_e) + \sum_{w \in W}(1 - x_w) + \sum_{v \in f \setminus (U \cup W)} (2 - 2x_v) \geq 0$$

$$2y_f - 1 + \sum_{u \in U}(1 - x_u) + (1 - y_g) + \sum_{v \in f \setminus (U \cup W)} (2 - 2x_v) \geq 0$$

$$2y_f - 1 + (1 - y_e) + (1 - y_g) + \sum_{v \in f \setminus (U \cup W)} (2 - 2x_v) \geq 0$$

$$x_u - 2y_e + x_w \geq 0$$

$$x_v - y_e \geq 0$$



Simple Odd β-Cycle Inequalities

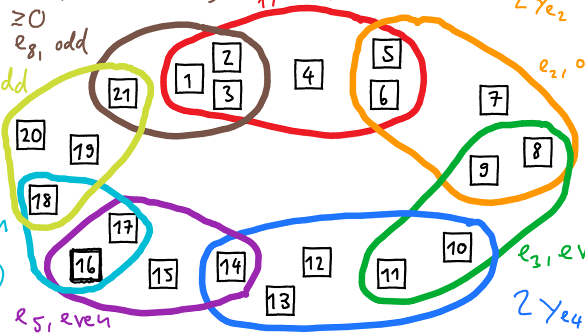
Example:

$2y_{e_1} - 1 + (1-x_1) + (1-x_2) + (1-x_3) + (2-2x_4) + (1-x_5) + (1-x_6) \ge 0$

$2y_{e_8} - 1 + (1-x_1) + (1-x_2) + (1-x_3) + (1-x_{21}) \ge 0$

$2y_{e_2} - 1 + (1-x_5) + (1-x_6) + (2-2x_7) + (1-y_{e_3}) \ge 0$

$2y_{e_7} - 1 + (1-x_{21}) + (2-2x_{19}) + (2-2x_{20}) + (1-y_{e_6}) \ge 0$



0

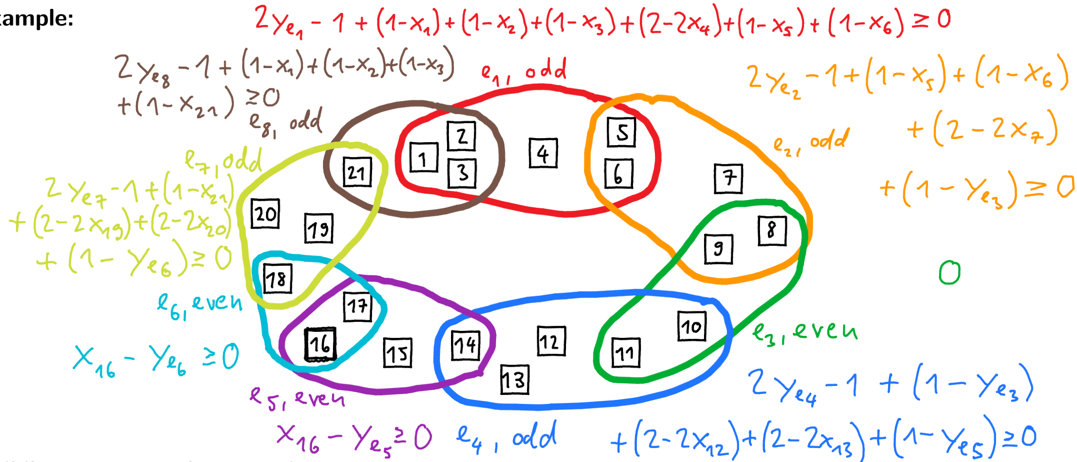
$x_{16} - y_{e_6} \ge 0$

$x_{16} - y_{e_5} \ge 0$

$2y_{e_4} - 1 + (1-y_{e_3}) + (2-2x_{12}) + (2-2x_{13}) + (1-y_{e_5}) \ge 0$

Simple Odd β-Cycle Inequalities

Example:

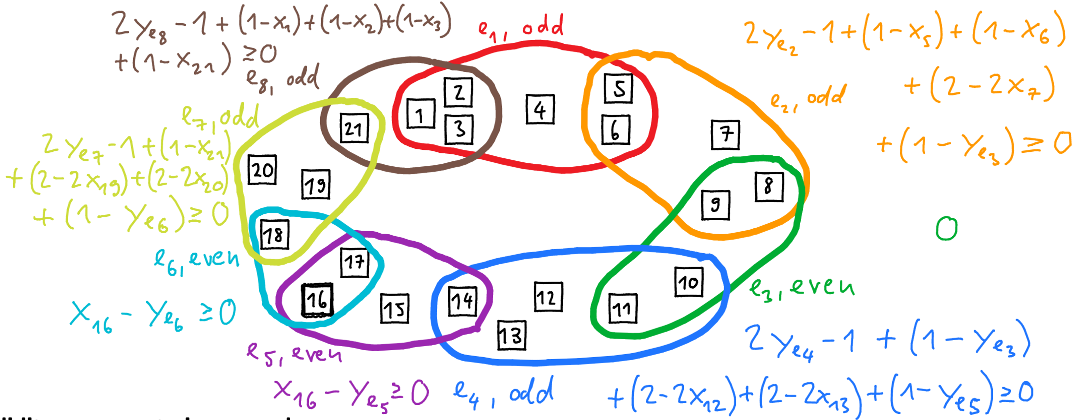


Validity arguments in general:

- Add building blocks along a cyclic walk such that overlapping terms add up to something even.

Simple Odd β-Cycle Inequalities

Example: $2y_{e_1} - 1 + (1-x_1) + (1-x_2) + (1-x_3) + (2-2x_4) + (1-x_5) + (1-x_6) \geq 0$

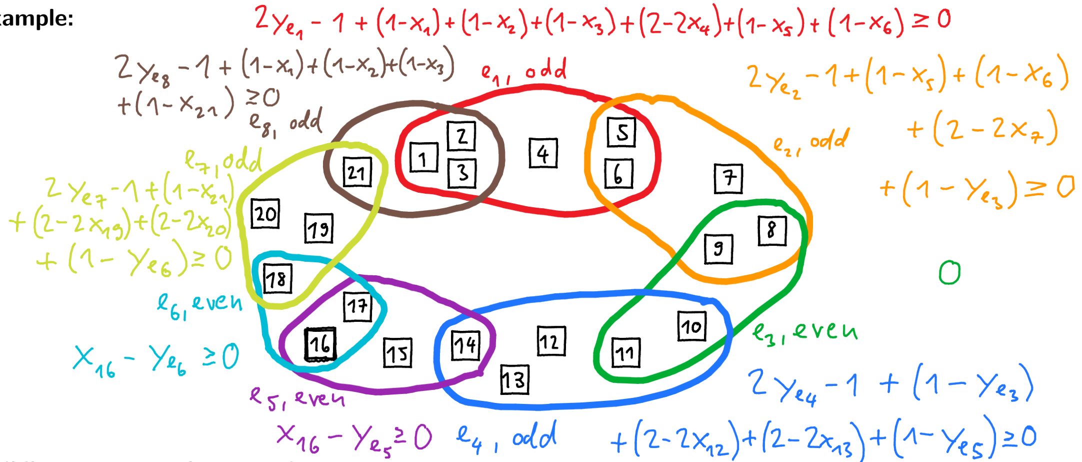


Validity arguments in general:

- Add building blocks along a cyclic walk such that overlapping terms add up to something even.
- This yields a new valid inequality with only <u>even</u> coefficients.

Simple Odd β-Cycle Inequalities

Example:



Validity arguments in general:

- Add building blocks along a cyclic walk such that overlapping terms add up to something even.
- This yields a new valid inequality with only even coefficients.
- If the first four building blocks occur an odd number of times, the right-hand side is odd.

Simple Odd β-Cycle Inequalities

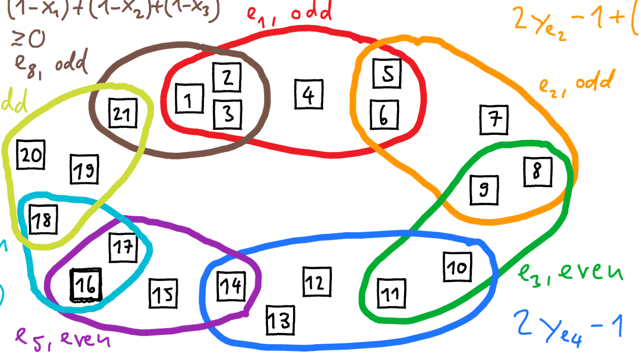
Example:

$$2y_{e_1} - 1 + (1-x_1) + (1-x_2) + (1-x_3) + (2-2x_4) + (1-x_5) + (1-x_6) \geq 0$$

$$2y_{e_8} - 1 + (1-x_1) + (1-x_2) + (1-x_3) + (1-x_{21}) \geq 0$$

$$2y_{e_2} - 1 + (1-x_5) + (1-x_6) + (2-2x_7) + (1-y_{e_3}) \geq 0$$

$$2y_{e_7} - 1 + (1-x_{21}) + (2-2x_{19}) + (2-2x_{20}) + (1-y_{e_6}) \geq 0$$



0

$$x_{16} - y_{e_6} \geq 0$$

$$x_{16} - y_{e_5} \geq 0$$

$$2y_{e_4} - 1 + (1-y_{e_3}) + (2-2x_{12}) + (2-2x_{13}) + (1-y_{e_5}) \geq 0$$

Validity arguments in general:

- Add building blocks along a cyclic walk such that overlapping terms add up to something even.
- This yields a new valid inequality with only even coefficients.
- If the first four building blocks occur an odd number of times, the right-hand side is odd.
- Hence, we can increase right-hand side by 1. (or: scale by $\frac{1}{2}$ and round rhs up)

Separation Algorithm

Auxiliary graph:

- Auxiliary nodes (c, p) where c is a "connection point" $(V \cup E \cup \{e \cap f : e, f \in E\})$ and $p \in \{\pm 1\}$ is a parity.
- Auxiliary edges from (c, p) to (c', p') for building block inequality $s(x, y) \geq 0$ between connection points c and c'. Parities $p \neq p'$ if and only if one of first four inequalities.
- Length of edge: slack $s(\hat{x}, \hat{y})$ for given vector $(\hat{x}, \hat{y})$.

Separation Algorithm

Auxiliary graph:

- Auxiliary nodes (c, p) where c is a "connection point" $(V \cup E \cup \{e \cap f : e, f \in E\})$ and $p \in \{\pm 1\}$ is a parity.
- Auxiliary edges from (c, p) to (c', p') for building block inequality $s(x, y) \geq 0$ between connection points c and c'. Parities $p \neq p'$ if and only if one of first four inequalities.
- Length of edge: slack $s(\hat{x}, \hat{y})$ for given vector $(\hat{x}, \hat{y})$.

Lemma – Reduction to shortest odd cycle problem **[Del Pia, Walter '22]**

Walks from (c, p) to $(c, -p)$ of length less than 1 $\longleftrightarrow$ simple odd β-cycle inequality violated by $(\hat{x}, \hat{y})$

Auxiliary graph:

- Auxiliary nodes (c, p) where c is a "connection point" $(V \cup E \cup \{e \cap f : e, f \in E\})$ and $p \in \{\pm 1\}$ is a parity.
- Auxiliary edges from (c, p) to (c', p') for building block inequality $s(x, y) \geq 0$ between connection points c and c'. Parities $p \neq p'$ if and only if one of first four inequalities.
- Length of edge: slack $s(\hat{x}, \hat{y})$ for given vector $(\hat{x}, \hat{y})$.

Lemma – Reduction to shortest odd cycle problem **[Del Pia, Walter '22]**

Walks from (c, p) to $(c, -p)$ of length less than 1 $\longleftrightarrow$ simple odd β-cycle inequality violated by $(\hat{x}, \hat{y})$

Consequence:

Theorem – Separation algorithm **[Del Pia, Walter '22]**

Let $G = (V, E)$ be a hypergraph and let $(\hat{x}, \hat{y}) \in \text{FR}(G)$. The separation problem for simple odd β-cycle inequalities can be solved in time $\mathcal{O}(|E|^5 + |V|^2 \cdot |E|)$.

Auxiliary graph:

- Auxiliary nodes (c, p) where c is a "connection point" $(V \cup E \cup \{e \cap f : e, f \in E\})$ and $p \in \{\pm 1\}$ is a parity.
- Auxiliary edges from (c, p) to (c', p') for building block inequality $s(x, y) \geq 0$ between connection points c and c'. Parities $p \neq p'$ if and only if one of first four inequalities.
- Length of edge: slack $s(\hat{x}, \hat{y})$ for given vector $(\hat{x}, \hat{y})$.

Lemma – Reduction to shortest odd cycle problem **[Del Pia, Walter '22]**

Walks from (c, p) to $(c, -p)$ of length less than 1 $\longleftrightarrow$ simple odd β-cycle inequality violated by $(\hat{x}, \hat{y})$

Consequence:

Theorem – Separation algorithm **[Del Pia, Walter '22]**

Let $G = (V, E)$ be a hypergraph and let $(\hat{x}, \hat{y}) \in \text{FR}(G)$. The separation problem for simple odd β-cycle inequalities can be solved in time $\mathcal{O}(|E|^5 + |V|^2 \cdot |E|)$.

Proof:

- By the lemma above, it suffices to run Dijkstra's algorithm once for per connection point. □

Preliminary Computational Results

Implementation:

- Plugin for SCIP solver framework that inspects all AND constraints and builds hypergraph G.
- Separation is done by increasing complexity:
  1. Violated inequalities from $\text{SR}(G)$.
  2. Violated 1-flower inequalities.
  3. Violated 2-flower inequalities.
  4. Violated simple odd β-cycle inequalities.

Preliminary Computational Results

Implementation:

- Plugin for SCIP solver framework that inspects all AND constraints and builds hypergraph G.
- Separation is done by increasing complexity:
  1. Violated inequalities from $\mathrm{SR}(G)$.
  2. Violated 1-flower inequalities.
  3. Violated 2-flower inequalities.
  4. Violated simple odd β-cycle inequalities.

Instances:

1. Image restoration instances from computer vision. [used by Crama, Rodríguez-Heck '17]
2. Low autocorrelated binary sequence problem from POLIP / MINLPLib benchmark libraries for polynomial / mixed-integer nonlinear optimization. [used by Del Pia, Di Gregorio '21]

- For both, the maximum polynomial degree is 4, i.e., $|e| \leq 4$ for all $e \in E$.

Preliminary Computational Results

Implementation:

- Plugin for SCIP solver framework that inspects all AND constraints and builds hypergraph G.
- Separation is done by increasing complexity:
  1. Violated inequalities from SR(G).
  2. Violated 1-flower inequalities.
  3. Violated 2-flower inequalities.
  4. Violated simple odd β-cycle inequalities.

Instances:

1. Image restoration instances from computer vision. [used by Crama, Rodríguez-Heck '17]
2. Low autocorrelated binary sequence problem from POLIP / MINLPLib benchmark libraries for polynomial / mixed-integer nonlinear optimization. [used by Del Pia, Di Gregorio '21]

- For both, the maximum polynomial degree is 4, i.e., $|e| \leq 4$ for all $e \in E$.

Experiment:

- Disable all other solver features, i.e., general-purpose cutting planes, presolve (except for linearization of the polynomial), symmetry breaking, restarts, heuristics.
- Compare obtained dual bounds to best known primal solution.
- Question to answer: **How much gap can these inequalities close?**

Image Restoration

Table: Remaining gap (in %) and computation time to compute the dual bounds of different relaxations.

Image	$\|V\|$	$\|E\|$	Standard		1-flower		2-flower		S. odd β-cycle		SCIP cuts	
10×10	100	534	69.30 %	0.1 s	10.33 %	0.3 s	10.33 %	0.3 s	0.0 %	1.9 s	18.69 %	1.5 s
10×15	150	838	43.84 %	0.3 s	8.85 %	0.7 s	8.85 %	0.7 s	0.0 %	3.3 s	12.87 %	1.9 s
15×15	225	1275	63.00 %	0.7 s	12.80 %	2.4 s	12.80 %	2.4 s	0.0 %	5.3 s	22.47 %	7.1 s
15×20	300	1731	38.75 %	1.1 s	0.0 %	1.1 s	0.0 %	1.1 s	0.0 %	1.4 s	8.56 %	4.8 s
20×20	400	2353	39.46 %	1.7 s	0.14 %	2.8 s	0.14 %	2.8 s	0.0 %	4.8 s	17.86 %	24.2 s
20×25	500	2978	41.48 %	3.2 s	3.86 %	3.9 s	3.86 %	3.9 s	0.11 %	15.1 s	23.19 %	16.0 s
25×25	625	3718	41.00 %	2.9 s	0.26 %	5.2 s	0.26 %	5.2 s	0.04 %	17.1 s	11.80 %	12.5 s

Remark:

- Shown are averages over the 15 instances for each image size.

Image Restoration

Table: Remaining gap (in %) and computation time to compute the dual bounds of different relaxations.

Image	$\|V\|$	$\|E\|$	Standard		1-flower		2-flower		S. odd β-cycle		SCIP cuts	
10×10	100	534	69.30 %	0.1 s	10.33 %	0.3 s	10.33 %	0.3 s	0.0 %	1.9 s	18.69 %	1.5 s
10×15	150	838	43.84 %	0.3 s	8.85 %	0.7 s	8.85 %	0.7 s	0.0 %	3.3 s	12.87 %	1.9 s
15×15	225	1275	63.00 %	0.7 s	12.80 %	2.4 s	12.80 %	2.4 s	0.0 %	5.3 s	22.47 %	7.1 s
15×20	300	1731	38.75 %	1.1 s	0.0 %	1.1 s	0.0 %	1.1 s	0.0 %	1.4 s	8.56 %	4.8 s
20×20	400	2353	39.46 %	1.7 s	0.14 %	2.8 s	0.14 %	2.8 s	0.0 %	4.8 s	17.86 %	24.2 s
20×25	500	2978	41.48 %	3.2 s	3.86 %	3.9 s	3.86 %	3.9 s	0.11 %	15.1 s	23.19 %	16.0 s
25×25	625	3718	41.00 %	2.9 s	0.26 %	5.2 s	0.26 %	5.2 s	0.04 %	17.1 s	11.80 %	12.5 s

Remark:

- Shown are averages over the 15 instances for each image size.

Observations – Image Restoration Instances **[Del Pia, Walter '22]**

- 1-flower inequalities close a lot of gap already.
- 2-flower inequalities were never violated after adding 1-flowers.
- Simple odd β-cycle inequalities close almost all of the remaining gap.
- General-purpose cutting planes of SCIP are outperformed.

Low Autocorrelated Binary Sequences (1)

Table: Remaining gap (in %) and computation time to compute the dual bounds of different relaxations.

Instance	$\|V\|$	$\|E\|$	Standard		1-flower = 2-flower		S. odd β-cycle		SCIP cuts	
20-05	20	187	884.62 %	0.0 s	310.10 %	0.0 s	228.37 %	0.2 s	458.65 %	1.1 s
20-10	20	813	1428.61 %	0.2 s	519.41 %	0.6 s	365.8 %	59.0 s	1174.08 %	0.9 s
20-15	20	1474	1564.03 %	0.8 s	570.87 %	2.1 s	405.3 %	367.0 s	1374.16 %	1.4 s
25-06	25	382	1116.67 %	0.1 s	400.00 %	0.2 s	276.04 %	2.2 s	760.52 %	0.9 s
25-13	25	1757	1518.46 %	0.9 s	553.51 %	2.2 s	391.1 %	680.0 s	1344.46 %	2.3 s
25-19	25	3015	1645.87 %	2.3 s	602.50 %	6.9 s	$\leq$ 428.67 %	> 1 h	1518.22 %	3.8 s
25-25	25	3652	1730.76 %	3.0 s	633.80 %	9.0 s	$\leq$ 454.81 %	> 1 h	1573.28 %	6.8 s
30-04	30	193	633.33 %	0.0 s	211.11 %	0 s	211.11 %	0.0 s	223.46 %	1.2 s
30-08	30	896	1308.67 %	0.2 s	473.44 %	0.5 s	330.79 %	15.4 s	1035.81 %	1.1 s
30-15	30	2914	1598.37 %	1.2 s	584.75 %	4.1 s	414.39 %	2357.0 s	1428.35 %	4.5 s
30-23	30	5346	1717.31 %	3.5 s	630.39 %	11.8 s	$\leq$ 468.46 %	> 1 h	1634.17 %	7.6 s
30-30	30	6382	1782.08 %	4.8 s	653.86 %	18.1 s	$\leq$ 576.22 %	> 1 h	1696.71 %	14.3 s
35-04	35	228	633.33 %	0.0 s	210.94 %	0 s	210.94 %	0.0 s	238.28 %	1.2 s
35-09	35	1346	1417.93 %	0.4 s	516.48 %	1.0 s	362.92 %	59.1 s	1164.69 %	2.1 s
35-18	35	4967	1652.86 %	3.3 s	605.63 %	10.9 s	$\leq$ 436.68 %	> 1 h	1577.31 %	8.0 s
35-26	35	8312	1738.75 %	9.1 s	638.56 %	36.4 s	$\leq$ 543.43 %	> 1 h	1666.42 %	19.0 s
40-05	40	407	884.62 %	0.0 s	310.26 %	0.1 s	228.21 %	0 s	505.98 %	1.8 s
40-10	40	2013	1498.58 %	0.6 s	547.79 %	2.0 s	386.3 %	139.0 s	1289.21 %	3.3 s
40-30	40	7203	1790.20 %	21.1 s	658.76 %	94.1 s	unknown	> 1 h	1757.06 %	42.5 s
40-40	40	15344	2246.31 %	33.7 s	839.29 %	202.0 s	unknown	> 1 h	2195.81 %	65.4 s

Low Autocorrelated Binary Sequences (2)

Table: Remaining gap (in %) and computation time to compute the dual bounds of different relaxations.

Instance	$\|V\|$	$\|E\|$	Standard		1-flower = 2-flower		S. odd β-cycle		SCIP cuts	
45-05	45	462	882.77 %	0.1 s	309.46 %	0 s	227.62 %	0 s	514.98 %	2.9 s
45-11	45	2768	1558.63 %	1.1 s	571.21 %	3.6 s	403.78 %	392.0 s	1415.96 %	5.9 s
45-23	45	10731	1790.52 %	14.4 s	659.88 %	63.6 s	$\leq$ 575.22 %	$>$ 1 h	1754.46 %	34.7 s
45-34	45	18303	3270.63 %	47.1 s	1252.4 %	348.0 s	unknown	$>$ 1 h	3216.41 %	94.2 s
45-45	45	21948	54 456.45 %	94.0 s	21 736.22 %	647.0 s	unknown	$>$ 1 h	53 662.93 %	119.0 s
50-06	50	832	1116.67 %	0.2 s	400.00 %	0.3 s	275.46 %	3.9 s	794.72 %	1.8 s
50-13	50	4407	1616.87 %	2.4 s	593.25 %	8.6 s	420.29 %	3131.0 s	1499.09 %	8.0 s
50-25	50	14362	2133.10 %	26.3 s	797.19 %	195.0 s	unknown	$>$ 1 h	2094.17 %	40.9 s
50-38	50	25396	39 284.44 %	107.0 s	15 696.67 %	1241.0 s	unknown	$>$ 1 h	38 675.21 %	219.0 s
50-50	50	30221	65 563.27 %	163.0 s	26 178.91 %	1424.0 s	unknown	$>$ 1 h	64 675.26 %	206.0 s
55-06	55	922	1124.83 %	0.2 s	403.36 %	0.4 s	277.94 %	4.3 s	804.18 %	2.1 s
55-14	55	5735	1687.63 %	4.2 s	621.19 %	15.3 s	$\leq$ 441.43 %	$>$ 1 h	1608.75 %	10.3 s
55-28	55	19592	12 541.35 %	63.7 s	7270.68 %	231.0 s	unknown	$>$ 1 h	12 352.26 %	96.9 s
55-41	55	33013	49 516.78 %	189.0 s	$\leq$ 26 840.37 %	$>$ 1 h	unknown	$>$ 1 h	48 682.01 %	403.0 s
55-55	55	40087	77 649.06 %	320.0 s	$\leq$ 35 170.20 %	$>$ 1 h	unknown	$>$ 1 h	76 351.19 %	507.0 s
60-08	60	1976	1409.51 %	0.6 s	514.49 %	1.7 s	361.20 %	48.3 s	1227.38 %	3.7 s
60-15	60	7234	1662.86 %	6.8 s	610.75 %	27.5 s	$\leq$ 435.55 %	$>$ 1 h	1610.77 %	15.9 s
60-30	60	24720	147 684.48 %	116.0 s	97 525.37 %	1685.0 s	unknown	$>$ 1 h	145 471.15 %	234.0 s
60-45	60	43129	58 469.68 %	350.0 s	32 242.86 %	2186.0 s	unknown	$>$ 1 h	57 950.54 %	458.0 s
60-60	60	51970	94 731.68 %	570.0 s	$\leq$ 75 083.58 %	$>$ 1 h	unknown	$>$ 1 h	93 319.83 %	$>$ 1 h

Observations – Low Autocorrelated Binary Sequences [Del Pia, Walter '22]

- 1-flower inequalities close a lot of gap already.
- 2-flower inequalities were never violated after adding 1-flowers.
- General-purpose cutting planes of SCIP are outperformed.

Low Autocorrelated Binary Sequences (3)

Observations – Low Autocorrelated Binary Sequences **[Del Pia, Walter '22]**

- Extremely large gaps in general.
- 1-flower inequalities close a lot of gap already.
- 2-flower inequalities were never violated after adding 1-flowers.
- General-purpose cutting planes of SCIP are outperformed.

Low Autocorrelated Binary Sequences (3)

Observations – Low Autocorrelated Binary Sequences **[Del Pia, Walter '22]**

- Extremely large gaps in general.
- Hypergraphs are quite dense.
- 1-flower inequalities close a lot of gap already.
- 2-flower inequalities were never violated after adding 1-flowers.
- Simple odd β-cycle inequalities are (currently) too expensive.
- General-purpose cutting planes of SCIP are outperformed.

Low Autocorrelated Binary Sequences (3)

Observations – Low Autocorrelated Binary Sequences **[Del Pia, Walter '22]**

- Extremely large gaps in general.
- Hypergraphs are quite dense.
- 1-flower inequalities close a lot of gap already.
- 2-flower inequalities were never violated after adding 1-flowers.
- Simple odd β-cycle inequalities are (currently) too expensive.
- General-purpose cutting planes of SCIP are outperformed.

Remark:

- Implementation of 2-flower separation was carefully checked for correctness :-)

Conclusions & Future Work

Conclusions – Future Research Directions [Del Pia, Walter '22]

- New inequalities are the right relaxation for separation.
- Nice theoretical properties remain!
- Strengthening of generated inequalities should be possible.
- Auxiliary graph is of polynomial size, but $\mathcal{O}(|E|^2 + |V|)$ nodes and $\mathcal{O}(|E|^3 + |E| \cdot |V|)$ edges is not exactly small in practice.

Conclusions & Future Work

Conclusions – Future Research Directions [Del Pia, Walter '22]

- New inequalities are the right relaxation for separation.
- Nice theoretical properties remain!
- Strengthening of generated inequalities should be possible.
- Auxiliary graph is of polynomial size, but $\mathcal{O}(|E|^2 + |V|)$ nodes and $\mathcal{O}(|E|^3 + |E| \cdot |V|)$ edges is not exactly small in practice.

Good news:

- Preliminary results on using a smaller auxiliary graph $\rightsquigarrow$ expect drastic reduction of running time.

Conclusions – Future Research Directions **[Del Pia, Walter '22]**

- New inequalities are the right relaxation for separation.
- Nice theoretical properties remain!
- Strengthening of generated inequalities should be possible.
- Auxiliary graph is of polynomial size, but $\mathcal{O}(|E|^2 + |V|)$ nodes and $\mathcal{O}(|E|^3 + |E| \cdot |V|)$ edges is not exactly small in practice.

Good news:

- Preliminary results on using a smaller auxiliary graph $\rightsquigarrow$ expect drastic reduction of running time.

Happy Birthday SCIP!